Week 4 - Friday

COMP 2230

Last time

- Rational numbers
- Divisibility

Questions?

Assignment 2

Logical warmup

- Find a number with the letters of its English name in alphabetical order.

Sequences

Seek whence

- Mathematicians love patterns
- Perhaps the best example of that is **sequences**
- A sequence is a set of elements, usually numbers, written in order
- Each element is called a **term**
- We can number each term, calling that number the **subscript** or **index**
- Sequences can be finite or infinite

Examples

- What's the next term in each sequence?
- 1, 2, 3, 4, ...
- 1, 4, 9, 16, ...
- 1, 1, 2, 3, 5, ...
- 1, 2, ...
  - Wrong! The next term is 720!. Yes, that's actually factorial. What's the term after that?

Explicit formulas

- It is often possible to use a formula to describe the relationship between the value of a subscript and the value of the corresponding term
- Let term $a_k = \frac{k}{k+1}$, for integers $k \geq 1$
- Let term $b_i = \frac{i-1}{i}$, for integers $i \geq 2$
- What are the first few terms of each?

Alternating sequences

- It's important to note that not all sequences are monotonic
- Give an explicit formula for $1, -3, 5, -7, 9, \ldots$

Summation notation

- **Summation notation** is used to describe a summation of some part of a sequence

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \ldots + a_n$$

- **Expanded form** is the summation written without the sigma
- Compute the following:
- $\sum_{i=0}^{6} 2^i =$
- $\sum_{i=0}^{4} i^3 =$

Convert to summation notation

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \ \dots \ + \frac{n+1}{2n}$$

$$\frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ \dots \ + \frac{1}{n \cdot (n+1)}$$

Product notation

- **Product notation** is used to describe a product of some part of a sequence

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \ldots \cdot a_n$$

- **Expanded form** is the product written without the pi
- Compute the following:
- $\prod_{i=1}^{6} 2^i =$
- $\prod_{i=1}^{4} 5 - i =$

Factorial

- n factorial is written $n!$
- $n! = n \cdot (n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$
- We define $0!$ to be 1, for reasons that may become clear later
- Factorial grows fast, even faster than 2^n
- The growth rate of factorial is roughly $2^{n \log n}$
- Write factorial using product notation

Properties of sums and products

- Be careful about moving things in and out of products and sums
- The following are allowed:

1. $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(a_k + b_k)$
2. $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$
3. $(\prod_{k=m}^{n} a_k) \cdot (\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n}(a_k \cdot b_k)$

Changing variables

- Changing from one dummy variable to another is always allowed, provided that all occurrences are changed
- Changing the bounds of a variable is allowed, provided that the summation is suitably adjusted
- Example:

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j}$$

Mathematical Induction

Induction

- General **induction** is moving from a specific set of facts to a general conclusion
- Example:
  - There are no tigers here.
  - I have a rock in my pocket.
  - Conclusion: My rock keeps tigers away.
- Induction can lead you to invalid conclusions
- So far in this class, we have only used **deduction**, which reasons from general truths to a specific conclusion

Mathematical induction

- Mathematical induction is special
- First, we need a property $P(n)$ that's defined for integers n
- Then, we need to know that it's true for some specific $P(a)$
- Then, we try to show that for all integers $k \geq a$, if $P(k)$ is true, it must be the case that $P(k+1)$ is true
- If we do that, $P(n)$ is true for all integers $n \geq a$
- Why?

Proof by mathematical induction

- To prove a statement of the following form:
  - $\forall n \in \mathbb{Z}$, where $n \geq a$, property $P(n)$ is true
- Use the following steps:

1. Basis Step: Show that the property is true for $P(a)$
2. Induction Step:
   - Suppose that the property is true for some $n = k$, where $k \in \mathbb{Z}, k \geq a$
   - Now, show that, with that assumption, the property is also true for $k + 1$

Example

- Prove that, for all integers $n \geq 1$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- **Hint:** Use induction

Example

- Prove that, for all integers $n \geq 0$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

- **Hint:** Use induction

Example

- Prove that, for all integers $n \geq 8$, $n = 3a + 5b$, where a,b are integers greater than or equal to zero
- **Hint:** Use induction and cases
- This is the example given in the book
- Another way to write it is that any amount of change above 8 cents can be made using only 3 cent and 5 cent coins

Upcoming

Next time…

- Geometric series
- Strong induction
- Examples

Reminders

- Keep working on Assignment 2
- Read 5.2, 5.3, and 5.4